GENERATING FAMILIES AND CONSTRUCTIBLE SHEAVES

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ABSTRACT. Let Λ be a Legendrian in the jet space of some manifold X. To a generating family presentation of Λ, we associate a constructible sheaf on $X \times \mathbb{R}$ whose singular support at infinity is Λ, and such that the generating family homology is canonically isomorphic to the endomorphism algebra of this sheaf. That is, the theory of generating family homology embeds in sheaf theory, and more specifically in the category studied in [STZ]. When $X = \mathbb{R}$, i.e., for the theory of Legendrian knots and links in the standard contact $\mathbb{R}^3$, we use ideas from the proof of the h-cobordism theorem to show this embedding is an equivalence. Combined with the results of [NRSSZ], this implies in particular that the generating family homologies of a knot are the same as its linearized Legendrian contact homologies.

1. INTRODUCTION

Generating families are by now a well-established tool in symplectic [LS, Sik, Vit, Vit2, EG, Cha, The] and contact [Che2, Pus, Tra2, FR, JT, San, HR, ST, SS, BST] topology. Their use has also been informed by the classical study of families of functions [Cer, HW, Wal].

A newer line of work exploits the microlocal study of constructible sheaves initiated by Kashiwara and Schapira [KS]. This theory was always symplectic in nature – microlocal geometry takes place in the cotangent bundle – but the connection to symplectic questions of current interest was perhaps first made clear in the work of Nadler and Zaslow, who showed the category of constructible sheaves is equivalent to the infinitesimally wrapped Fukaya category of the cotangent bundle [NZ, Nad]; these methods were subsequently applied to study mirror symmetry for toric varieties [FLTZ, FLTZ2]. In a different direction, Tamarkin [Tam] explained how to use the constructible sheaf category to prove non-displaceability results, and Guillermou, Kashiwara, and Schapira [GKS, GS, Gui, Gui2] have further developed this perspective. Sheaf techniques have also been applied to the study of Legendrian knots [STZ, NRSSZ, Gui3], their interactions with cluster algebra [STWZ, STW], and to ordinary knot theory [Sh].

My purpose here is to connect these stories in the setting of Legendrian submanifolds of jet bundles, and especially for Legendrian knots and links in $\mathbb{R}^3$.[1]

1.1. From Morse theory to sheaf theory.

Consider a diagram $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$, where the map $\phi : Y \to X$ is a fibration of smooth manifolds. We view the diagram as a family of functions from the fibers $f_x : Y_x := \phi^{-1}(x) \to \mathbb{R}$. Such families of functions featured prominently in the works

[1] As far as I can tell, *all known applications of generating families in symplectic and contact topology factor through constructible sheaves*. Possibly a similar viewpoint underlies [Vit3].

of Smale [Sma1, Sma2], [Cer], and Hatcher and Wagoner [HW] on h-cobordism, s-cobordism, pseudo-isotopy, etc.

In this context, it is natural to collect on $X \times \mathbb{R}$ the fiberwise critical values, i.e., the pairs (x, r) such that $f_x : Y_x \to \mathbb{R}$ has a critical value at r. This is called the *Cerf graphic*. We denote it Φ_f. We can view Φ_f as the front projection of a Legendrian $\Lambda_f \subset J^1(X)$, which is smooth under certain genericity assumptions on f. This Legendrian is said to be generated by f.

One way to describe the basic idea of this paper is as follows: we propose to further decorate the Cerf graphic by the sublevel set cohomologies of the family of functions f_x. This data naturally glues into a constructible sheaf on $X \times \mathbb{R}$, which remembers all the cohomological information of the generating family.

This is philosophically similar to the "Morse complex sequences" used in [Pus2, Hen, HR, HR2]; the difference being that we discard all additional (e.g., metric-dependent) Morse-theoretic information. In fact, the sheaf theory never sees this data in the first place. This may seem a defect; the corresponding virtue is that the sheaf is naturally defined over the entirety of $X \times \mathbb{R}$, i.e. including points where the function f_x fails to be Morse. In fact, we need not impose *any* genericity assumptions on f.

Let us first consider the case when X is a point, $f : Y \to \mathbb{R}$ is a Morse function, and we have chosen an appropriate metric on Y. Then we can form the complex $Morse(Y, f; \Bbbk)$, where the generators are named by the critical points of f, and the differential counts gradient trajectories between critical points. We write $\Bbbk$ for whatever choice of coefficient ring we have made. This complex computes the homology or cohomology of M.

Recording the critical values gives an $\mathbb{R}$-filtration on the Morse complex. The subquotients of this filtration are the relative cohomologies of sublevel sets. We write $Morse_{<z}(Y, f; \Bbbk)$ for the subcomplex generated by critical points of critical value less than z. Assuming that the function f is (locally) of finite type, the filtration has (locally) finitely many steps.

An $\mathbb{R}$-filtered complex with (locally) finitely many steps gives rise, in general, to a (locally) constructible sheaf of complexes on $\mathbb{R}$.

For the theory of sheaves on manifolds, we refer to [KS]. Informally, a constructible sheaf of cochain complexes $\mathcal{F}$ on a space Z is a family of complexes $\mathcal{F}_z$ parameterized by the points z of Z; these are called the stalks of the sheaf. They should vary continuously, in the sense that for each $z \in Z$ there is some small ball $B_{\epsilon(z)}(z)$ so that, for any $z' \in B_{\epsilon(z)}(z)$, there is a morphism $\mathcal{F}_z \to \mathcal{F}_{z'}$. All such diagrams should commute in the homotopically appropriate sense. Finally, there should exist a stratification of Z such that morphisms $\mathcal{F}_z \to \mathcal{F}_{z'}$ are quasi-isomorphisms except when z and z' lie on different strata. Such sheaves form a category, which we denote by $Sh(Z)$, or by $Sh(Z, \Bbbk)$ when we want to emphasize that we take our complexes to be $\Bbbk$-complexes.

We write $\widetilde{Morse}(Y, f; \Bbbk)$ for the sheaf on $\mathbb{R}$ associated to the filtered Morse complex; its stalks are given by

$$\widetilde{Morse}(Y, f; \Bbbk)_z := Morse_{<z}(Y, f; \Bbbk)$$

To completely describe the constructible sheaf $\widetilde{Morse}(Y, f; \Bbbk)$, it suffices to give in addition the "generization maps" from the stalk at z to the stalks at $z \pm \epsilon$; on in other words, the restriction maps from small balls $B_\delta(z) \to B_{\delta'}(z \pm \epsilon)$, where $1 \gg \delta \gg \epsilon \gg \delta'$. These are given as follows. The map $\widetilde{Morse}(Y, f; \Bbbk)_z \to \widetilde{Morse}(Y, f; \Bbbk)_{z+\epsilon}$ is the natural inclusion of Morse complexes, and our definitions (and the locally finite type assumption on f) ensure that, for sufficiently small ϵ, the inclusion $\widetilde{Morse}(Y, f; \Bbbk)_{z-\epsilon} \to \widetilde{Morse}(Y, f; \Bbbk)_z$ is an *equality*. Thus we can define the generization map going the other direction by inverting this equality.

Up to quasi-isomorphism, $\widetilde{Morse}(Y, f; \Bbbk)$ is just recording the relative cohomologies of sublevel sets. It thus carries no information about the Morse function — we could have built it with singular cohomology instead.

The machinery of sheaf theory provides an elegant way to do this. Specifically, given any map of manifolds $s : A \to B$, there are "star" and "shriek" "push-forward" and "pull-back" morphisms induced between $Sh(A)$ and $Sh(B)$. The star pullback $s^* : Sh(B) \to Sh(A)$ is given on stalks by $(s^*\mathcal{F})_a = \mathcal{F}_{s(a)}$. The shriek pushforward $s_! : Sh(A) \to Sh(B)$ is given on stalks by $s_!(\mathcal{G})_b = H^*_c(s^{-1}(b), \mathcal{G})$. In this notation we allow ourselves a standard abuse of writing H^* when we really mean the chain complex, and $=$ when we mean quasi-isomorphic by a canonical quasi-isomorphism. The other morphisms s_* and $s^!$, are the left adjoints of s^* and $s_!$, respectively.

Denote the graph of f, i.e. the pairs $(y, f(y))$, by $\Gamma_f \subset Y \times \mathbb{R}$. We can consider the region below the graph as a universal sublevel set $U_f := \{(y, z) \mid f(y) < z\}$. We write $u_f : U_f \to \mathbb{R}$ for the projection to the second factor.

To access the relative cohomology measured by the Morse complex, it is convenient to compactify $f : Y \to \mathbb{R}$ to $\overline{f} : \overline{Y} \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is a closed interval, $\overline{Y}$ is a manifold with boundary, and $\overline{f}$ is trivialized on a collar neighborhood of the boundary; this is possible assuming f is finite type. We write $\underline{u}_f : \underline{U}_f \to \overline{\mathbb{R}}$ for the corresponding universal sublevel set. By Morse theory:

$$\underline{u}_{f!}\Bbbk \cong \widetilde{Morse}(Y, f; \Bbbk)$$

Indeed, at a stalk z, we are asserting $H^*_c(f^{-1}([-\infty, z)), \Bbbk) \cong Morse_{<z}(Y, f; \Bbbk)$, so all we are saying is that the Morse complex computes (relative) cohomology.

That is, one can view $\underline{u}_{f!}\Bbbk$ as a sort of promissory note for Morse theoretic calculations: without doing the Morse theory, we might not know what $\underline{u}_{f!}\Bbbk$ actually is, but nonetheless this expression faithfully encapsulates our ignorance. More to the point, it can be manipulated via the formalism of sheaf theory.

1.2. Constructible sheaves from generating families.

Definition 1. A generating family is a diagram $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$, where ϕ is a fibration. We require in addition that $\phi \times f$ is a fibration over the complement of a compact subset of $X \times \mathbb{R}$, and that moreover the above diagram admits a fibrewise compactification $X \xleftarrow{\overline{\phi}} \overline{Y} \xrightarrow{\overline{f}} \overline{\mathbb{R}}$, compatible with the fibration-at-infinity structure of $\phi \times f$ and hence unique, where $\overline{Y}$ is a manifold with boundary, $\overline{\mathbb{R}} = [-\infty, \infty]$ is a closed interval, and $\overline{f}$ is a map of manifolds with boundary.

We write $\Gamma_f \subset Y \times \mathbb{R}$ for the graph of f, and denote the region beneath it by

$$U_f := \{(y,z) \mid f(y) < z\} \subset Y \times \mathbb{R}$$

We also write $u_f : U_f \to X \times \mathbb{R}$ for the restriction of $\phi \times 1_{\mathbb{R}}$ to U_f. We denote by $\underline{u}_f : \underline{U}_f \to X \times \overline{\mathbb{R}}$ the compactification.

For a generating family in the above sense, we can consider the sheaf $\underline{u}_{f!}\Bbbk$. This is our additional decoration of the Cerf diagram. At stalks, it is

$$\underline{u}_{f!}\Bbbk|_{(x,z)} \cong H^*_c(f_x^{-1}([-\infty, z)); \Bbbk)$$

Thus the sheaf $\underline{u}_{f!}\Bbbk$ can be viewed as a way of organizing the would-be Morse theory of the f_x. At least for generic f and generic $x \in X$, the function f_x really is Morse, so on this locus we can understand $\underline{u}_{f!}\Bbbk$ Morse-theoretically. For x outside a codimension two subset of X, or in other words for one-parameter families of functions, we can appeal to Morse-Cerf theory to understand $\underline{u}_{f!}\Bbbk$. As we will explain in Section 4, doing so explicitly is the essential content of the Morse complex sequences of Henry and Rutherford [Hen, HR].

Remark. There are various reasons to prefer $\underline{u}_{f!}\Bbbk$ to $u_{f!}\Bbbk$, which all have to do with the fact whereas $u_{f!}\Bbbk$ records the compactly supported cohomology of the sub-level sets, $\underline{u}_{f!}\Bbbk$ records the compactly supported cohomology relative to the level set at $-\infty$. In particular, the stalks of $\underline{u}_{f!}\Bbbk$ in the region $X \times [-\infty, N)$ are acyclic for $N \ll 0$.

Alternatively, we could declare that we work in the category of sheaves (dg) quotiented by the local systems, in which $\underline{u}_{f!}\Bbbk \cong u_{f!}\Bbbk$. But by [Kel, Dri], it is the same to work with the sheaves orthogonal to local systems, which $\underline{u}_{f!}\Bbbk$ is, and $u_{f!}\Bbbk$ is generally not.

A fundamental observation, made already in [Vit3], is that the Legendrian Λ_f generated by f can be recovered from the *microsupport* of the sheaf $\underline{u}_{f!}\Bbbk$. The notion of microsupport of a sheaf was introduced by Kashiwara and Schapira and systematically developed in [KS]. Informally, the microsupport of a constructible sheaf $\mathcal{F}$ on Z is a conical Lagrangian $ss(\mathcal{F}) \subset T^*Z$ which collects the co-directions along which the generization maps $\mathcal{F}_z \to \mathcal{F}_{z'}$ are nontrivial. In particular it is contained in the union of the conormal bundles to the strata along which the generization maps are (quasi-)isomorphisms.

There is a natural inclusion of the jet bundle $J^1(X) \to T^*(X \times \mathbb{R})$, in terms of which:

Theorem 2. *Let $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ be a generating family, which generates the Legendrian Λ_f. Then the microsupport of $\underline{u}_{f!}\Bbbk$ is the union of the zero section and the cone over Λ_f.*

We recall the definitions of the above notions and prove this theorem in Section 2.

Classically, one considers the set of generating families which generate a given Legendrian. By [LS, Vit], Legendrian isotopies induce bijections of this set, respecting certain invariants. As we will show here, those invariants can be recovered from $\underline{u}_{f!}\Bbbk$.

This (retroactively) motivates a newer approach to Legendrian invariants, introduced in [STZ], and compared to holomorphic curve approaches in [NRSSZ]. The idea is to study the category of $Sh_{\Lambda_f}(X \times \mathbb{R}; \mathbb{k})_0$ of sheaves microsupported in Λ_f as above. This category can also be shown to be a Legendrian invariant, via the sheaf quantization theorem of [GKS]. The theorem asserts that a generating function gives rise to an object of this category, i.e., $\underline{u}_{f!}\mathbb{k} \in Sh_{\Lambda_f}(X \times \mathbb{R}; \mathbb{k})_0$.

1.3. **A category of generating families.** Note that Sh is a *category*, i.e., there is a notion of morphisms between two sheaves. Our first new result is the identification of the classical name of the space of endomorphisms:

Theorem 3. *Let f be a generating family. Then $\mathrm{Hom}_{Sh}(\underline{u}_{f!}\mathbb{k}, \underline{u}_{f!}\mathbb{k})$ is the generating family homology of f. In particular, the generating family homology carries the structure of a unital ring.*

We recall the definition of generating family homology from [Tra2, FR] and prove Theorem 3 in Section 3.

This result allows us to import the entire categorical framework of sheaves into the world of generating families:

Definition 4. We write $Gen(X; \mathbb{k})$ for the (dg) category whose objects are generating families $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$, and whose morphisms and compositions are determined by requiring $f \mapsto \underline{u}_{f!}\mathbb{k}$ to be fully faithful.

For $\Lambda \subset J^1(X)$ a Legendrian submanifold, we write $Gen_\Lambda(X; \mathbb{k})$ for the full subcategory on generating families which generate Λ.

That is, we have set up a category of generating families, whose endomorphisms are generating family homology. Note in particular that, since we have associated to each generating family for Λ a sheaf *on the same space* $X \times \mathbb{R}$, we can say what it means for generating families $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ and $X \xleftarrow{\phi'} Y' \xrightarrow{f'} \mathbb{R}$ to be isomorphic, even though Y, Y' may be different spaces. This is a good notion: if f, f' are isomorphic in $Gen_\Lambda(X; \mathbb{k})$, then they have the same generating family cohomology, since Hom in any category is functorial in its components.

Remark. Classically, generating families were considered equivalent if they can be identified after stabilizing the bundle and pulling back by a fiberwise diffeomorphism. Such an equivalence certainly gives rise to an isomorphism in the present sense; we do not know to what extent the converse holds.

Theorem 3 and Definition 4 give a fully faithful morphism of (dg) categories $Gen_\Lambda(X; \mathbb{k}) \to Sh_\Lambda(X \times \mathbb{R}; \mathbb{k})_0$. We now set about studying its image.

Recall that the microsupport is the locus of codirections along which $\mathcal{F}_z \to \mathcal{F}_{z'}$ is not an isomorphism. There is a corresponding notion of microstalk, recording the cone of this morphism.

Let f be a generating family. Assuming that f_x is Morse for generic x, then the microstalks of $\underline{u}_{f!}\mathbb{k}$ will be shifted rank one $\mathbb{k}$-modules on the smooth locus of Λ_f. Assuming, as we do

henceforth, that Λ_f is smooth, the cohomological degree of this module – i.e. the Morse indices of the f_x – will determine a Maslov potential μ on Λ_f. We will write $Gen_{\Lambda,\mu}(X;\Bbbk)$ for the generating functions with this collection of Morse indices.

We write $Sh_{\Lambda;\mu}(X \times \mathbb{R};\Bbbk)_0 \subset Sh_\Lambda(X \times \mathbb{R},\Bbbk)$ for the full subcategory of sheaves with microstalks prescribed by μ, and acyclic stalks at $-\infty$.[2] Microstalks measure Morse indices, so $Gen_{\Lambda,\mu}(X;\Bbbk) \to Sh_{\Lambda;\mu}(X \times \mathbb{R};\Bbbk)_0$.

When $X = \mathbb{R}_x$, we can understand this morphism completely. We factor the map $f \mapsto \underline{u}_{f!}\Bbbk$ through a category of Morse complex sequences, which we introduced in [NRSSZ]; it is a categorical interpretation of the Morse complex sequences of [Pus2, Hen, HR, HR2]. The Morse complex sequences play two natural roles – first, any generalized Morse family of functions gives a Morse complex sequence, and second, any Morse complex sequence determines a sheaf. We recall the definition of this category in Section 4.2.

By adapting the proof of the h-cobordism theorem, we show:

Theorem 5. *Let M be a simply connected manifold. Let S be a Morse complex sequence over $\mathbb{Z}$ with all Morse indices satisfying $2 \leq \mu \leq \dim M - 2$. Assume given a Morse-Smale $f_{-\infty} : M \to \mathbb{R}_z$ whose corresponding filtered Morse complex is $S_{-\infty}$. Then there is a generating family $\mathbb{R}_x \leftarrow \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z$ extending $f_{-\infty}$, such that the corresponding Morse complex sequence is isomorphic to S.*

Remark. We define the notion of isomorphism of Morse complex sequences directly, but in fact this notion is determined by requiring that the natural map from Morse complex sequences to sheaves is fully faithful. The map from Morse complex sequences to sheaves is in turn essentially determined by requiring that it factor $f \mapsto \underline{u}_{f!}\Bbbk$.

In [NRSSZ], we showed that the Morse complex sequence category was isomorphic to the sheaf category over a field. We apply Theorem 5 to a linear map $f_{-\infty} : \mathbb{R}^N \to \mathbb{R}_z$ to conclude:

Corollary 6. *Let $\Lambda \subset J^1(\mathbb{R})$ be a Legendrian knot or link in $\mathbb{R}^3$. Assume that the Maslov potential $\mu \geq 2$. Then the morphism $Gen_{\Lambda,\mu}(X;\mathbb{Z}/p\mathbb{Z}) \to Sh_{\Lambda,\mu}(X \times \mathbb{R};\mathbb{Z}/p\mathbb{Z})_0$ is an equivalence for any prime p.*

In fact, it suffices to consider generating families of the form $\mathbb{R}_x \leftarrow \mathbb{R}_x \times \mathbb{R}^n \xrightarrow{f} \mathbb{R}_z$ where f_x is linear for $|x| \gg 0$ and f_x is linear at infinity for any x.

Remark. The assumption $\mu \geq 2$ above is harmless for practical purposes since shifting gives an equivalence of categories: $[k] : Sh_{\Lambda,\mu}(X \times \mathbb{R};\mathbb{Z}/p\mathbb{Z})_0 \to Sh_{\Lambda,\mu+k}(X \times \mathbb{R};\mathbb{Z}/p\mathbb{Z})_0$ for any k. In fact, this shift can be realized by an explicit Legendrian isotopy, see e.g. [NR].

In [NRSSZ], we defined an a (A_∞-)category $Aug_{\Lambda,\mu}(\Bbbk)$. Its objects are augmentations of the Chekanov-Eliashberg DGA [Che2, Eli], and its endomorphisms can be identified with linearized

[2] In [STZ, NRSSZ] we called this category $C_1(\Lambda,\mu;\Bbbk)$.

Legendrian contact homology. We proved this category is equivalent to the Morse complex sequence category over any ring $\Bbbk$. Thus:

Corollary 7. *For any Legendrian knot or link, the set of generating family homologies coincides with the set of linearized Legendrian contact homologies with $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ coefficients.*

Remark. The existence of a coincidence between generating family homologies and linearized Legendrian contact homologies was observed in some cases by Traynor [Tra2]. It was later shown by by Fuchs and Rutherford [FR] that the generating family homologies are a subset of the linearized contact homologies, and the above statement was conjectured in general. It is my understanding that Melvin, Sabloff, and Traynor also have an argument for the corollary.

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1.5. **Abuses of notation.** Throughout, all symbols should be understood to mean: the derived or dg construction most closely resembling whatever used to be meant by the symbol. All categories are dg categories. We use "$=$" to mean "isomorphic by an isomorphism so canonical as not to be worth mentioning." The pushforwards and pullbacks $\phi_*, \phi^*, \phi_!, \phi^!$ always mean the corresponding thing in the dg category of complexes. Finally, we write $\mathrm{H}^*(X;\Bbbk)$ or $\Gamma(X;\Bbbk)$ to mean a chain complex computing the cohomology of X with coefficients in $\Bbbk$.

2. The microsupport and the generated Legendrian

Here we prove Theorem 2. First we review the "correspondence" point of view on generating families from [EG]. Recall that a correspondence between manifolds is a submanifold of the product; for instance if $\phi : Y \to X$ is a map, then the graph $\Gamma_\phi \subset Y \times X$ gives a correspondence between Y and X. Passing to symplectic geometry, we get a correspondence between T^*Y and T^*X by taking the conormal to Γ_ϕ, which is canonically identified with $Y \times_X T^*X$:

$$T^*Y \xleftarrow{d\phi} Y \times_X T^*X \xrightarrow{\phi \times T^*X} T^*X$$

Convolution with this correspondence determines a push-forward of sets

$$\phi_\star := (\phi \times T^*X) \circ (d\phi)^{-1}$$

Now let $f : Y \to \mathbb{R}$ be a function. We write Γ_{df} for the graph of the differential. In the cases of interest, ϕ will be a fibration. We write

$$L_f := \phi_\star(\Gamma_{df}) \subset T^*X$$

Assuming certain genericity conditions, L_f is Lagrangian and is said to be generated by f; it collects the horizontal derivatives along the fibrewise critical points of the function f. From the definition,

$$\phi_\star(\Gamma_{df}) = \{(x, \xi \in T^*_x X) | \exists y, \phi(y) = x, df(y) = d\phi(\xi)\} \subset T^*X$$

Then the above says that we collect points y such that the image of $df(y)$ in the 'vertical' cotangent bundle $T^*\phi := T^*Y/\phi^*T^*X$ is zero. That is, y is a critical point of the restriction of f to the fibre of the fibration $Y \to X$ containing y. For such y, we can write $df(y) = \phi^*(x, \xi)$ for a unique (x, ξ); the collection of such (x, ξ) sweep out $\phi_\star(\Gamma_{df})$.

We write $\Phi_f \subset X \times \mathbb{R}$ for the fibrewise critical values, or in other words, the discriminant of $\phi \times f : Y \to X \times \mathbb{R}$.

The loci L_f and Φ_f are the Lagrangian and front projections of a Legendrian $\Lambda_f \subset J^1(X) = T^*X \times \mathbb{R}$ to its factors T^*X and $X \times \mathbb{R}$, respectively. This suffices to describe Λ_f, but for our purposes it is better to give it as a convolution, as before.

We identify

$$J^1(X) = T^*X \times \mathbb{R} \times \{-1\} \subset T^*(X \times \mathbb{R})$$

Here, the $\{-1\}$ is fixing the cotangent coordinate, not the base coordinate.

Let $\Gamma_f \subset Y \times \mathbb{R}$ be the graph of f. We take Λ_f to be the pushforward of the conormal bundle to the graph, restricted to the jet bundle of X:

$$\Lambda_f := (\phi \times 1_\mathbb{R})_\star(T^*_{\Gamma_f}(Y \times \mathbb{R})) \cap J^1(X)$$

To see the relation to Φ_f, note that the function $\phi \times f$ factors as the composition of $\phi \times 1_\mathbb{R} : Y \times \mathbb{R} \to X \times \mathbb{R}$ with the inclusion of the graph $Y \xrightarrow{\sim} \Gamma_f \hookrightarrow Y \times \mathbb{R}$.

The above passage from T^*X to $T^*(X \times \mathbb{R})$ trades exact immersed Lagrangians with vanishing Maslov class in T^*X for embedded conical Lagrangians in $T^*(X \times \mathbb{R})$. This trick was exploited to great effect in the context of sheaf theory by Tamarkin [Tam]. The simplest example of its use is the following: the image of $T^*_{\Gamma_f}(Y \times \mathbb{R}) \cap J^1(Y)$ in T^*Y is Γ_{df}.

We turn to the study of the microsupport of the sheaf $\underline{u}_! \mathbb{k}$. Recall that microsupport interacts well with pushforwards. Assuming that $\phi : B \to A$ is proper on the support of a sheaf $\mathcal{F}$ on B, we have [KS, Prop. 5.4.4]:

$$ss(\phi_! \mathcal{F}) \subset \phi_\star ss(\mathcal{F})$$

Here, $\phi_\star$ is the pushforward defined above by convolution.

To compute the microsupport of $\underline{u}_{f!}\mathbb{k}$, we factor $\underline{u}_f$ into $\underline{U}_f \xrightarrow{i} Y \times \overline{\mathbb{R}} \xrightarrow{\phi \times 1} X \times \overline{\mathbb{R}}$. Then:

$$\begin{aligned} ss(\underline{u}_{f!}\mathbb{k}) &= ss((\phi \times 1)_! i_! \mathbb{k}) \\ &\subset (\phi \times 1)_\star(\$ i_! \mathbb{k}) \\ &\subset (\phi \times 1)_\star\big(T^*_{\Gamma_f}(Y \times \overline{\mathbb{R}}) \cup T^*_{Y \times \overline{\mathbb{R}}}(Y \times \overline{\mathbb{R}})\big) \\ &\subset \overline{\mathbb{R}}\, \Lambda_f \cup T^*_{X \times \overline{\mathbb{R}}}(X \times \overline{\mathbb{R}}) \end{aligned}$$

Intersection with the jet bundle shows $ss(\underline{u}_{f!}\Bbbk) \subset \Lambda_f$. If the restriction of f to a generic fibre of $\phi : Y \to X$ is Morse, and Λ_f is a manifold, then it is easy to see that in fact the inclusion is an equality. We henceforth restrict to this case, and cease distinguishing the two notations for the generated Legendrian.

Since we used the ! pushforward, in fact the part of $ss(i_!\Bbbk)$ which lies away from the zero section is only the *negative* conormal to Γ_f. Following this through the convolution, $ss(\underline{u}_{f!}\Bbbk)$ lies in the negative half of the cotangent bundle, hence $ss(\underline{u}_{f!}\Bbbk) \subset \mathbb{R}_+\Lambda_f \cup T^*_{X\times\mathbb{R}}(X \times \mathbb{R})$.

Remark. We could now drop all assumptions on the diagram $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$, and declare that, in general, it generates the Legendrian $\Lambda_f := ss(\underline{u}_{f!}\Bbbk) \cap J^1(X)$.

3. Generating family cohomology is sheaf cohomology

Theorem 8. *If $X \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ and $X \xleftarrow{\phi'} Y' \xrightarrow{f'} \mathbb{R}$ are two generating families, we define the difference function*

$$\begin{aligned} w : Y \times_X Y' &\to \mathbb{R} \\ (x, y, y') &\mapsto f'(x, y') - f(x, y) \end{aligned}$$

Then

$$\mathrm{Hom}^*(\underline{u}_{f!}\Bbbk, \underline{u}_{f'!}\Bbbk) \cong \mathrm{H}^*(Y \times_X Y', w^{-1}(0,\infty); \Bbbk)[\dim Y - \dim X]$$

Proof. If $\mathcal{F}, \mathcal{G}$ are sheaves on some space B, and $\Delta_B : B \to B \times B$ is the diagonal, then $\mathcal{F} \otimes^! \mathcal{G} := \Delta_M^!(\mathcal{F} \boxtimes \mathcal{G})$. The sheaf Hom can be expressed in terms of this tensor product:

$$\mathcal{H}om(\underline{u}_{f!}\Bbbk, \underline{u}_{f'!}\Bbbk) = (\underline{u}_{f!}\Bbbk)^\vee \otimes^! \underline{u}_{f'!}\Bbbk = u_{f*}\Bbbk[\dim U_f] \otimes^! \underline{u}_{f'!}\Bbbk$$

We lost the underline in the final formula by Verdier duality on the manifold-with-boundary $\underline{U}_f$.

To use base change with respect to the diagonal, we have to interpret $\underline{u}_{f'!}$ in terms of $*$-pushforwards. Let $V_{f'} := \{(y, z) \mid f'(y) > z\} \subset Y' \times \mathbb{R}$ be the region *above* the graph of f', and $v_{f'} : V_f \to X \times \mathbb{R}$ the projection. We write $\overline{V}_{f'}$ to include $z = \infty$, and similarly $\underline{V}_{f'}$ to include the graph, i.e. to take $f'(y) \geq z$, and $\underline{\overline{V}}_{f'}$ to include both. We now have an open-closed decomposition of $\overline{Y}' \times \overline{\mathbb{R}}$ as

$$\underline{\overline{V}}_{f'} \xrightarrow{\bar{i}} \overline{Y}' \times \overline{\mathbb{R}} \xleftarrow{j} \underline{U}_{f'}$$

hence a triangle on $\overline{Y}' \times \overline{\mathbb{R}}$ of the form

$$\underline{j}_!\Bbbk \to \Bbbk \to \underline{\bar{i}}_*\Bbbk \xrightarrow{[1]}$$

We want to push this forward to $X \times \overline{\mathbb{R}}$ with $(\overline{\phi}' \times 1)_!$. Note that the map $\overline{\phi}' \times 1$ is proper on $\overline{Y}' \times \overline{\mathbb{R}}$ as well as on $\underline{\overline{V}}_{f'}$, so on the second and third terms, we can replace the ! with a $*$. (This last step is one place where it is important that we have taken $\underline{u}_{f!}$ rather than $u_{f!}$.) Finally, wherever we have $*$, we can drop the compactification of the source, since $*$-pushforward of the constant sheaf

to the compactification will in these cases be the constant sheaf on the compactification. Thus, on $X \times \overline{\mathbb{R}}$ we have a triangle of sheaves

$$\underline{u}_{f'!}\Bbbk \to (\phi' \times 1)_*\Bbbk \to v_{f'*}\Bbbk \xrightarrow{[1]}$$

and hence also

$$u_{f*}\Bbbk \otimes^! \underline{u}_{f'!}\Bbbk \to u_{f*}\Bbbk \otimes^! (\phi' \times 1)_*\Bbbk \to u_{f*}\Bbbk \otimes^! v_{f'*}\Bbbk \xrightarrow{[1]} \tag{1}$$

To analyse the second two terms, we use the fibre product diagrams

$$\begin{tikzcd}
Y \times_X Y' \times \mathbb{R} \arrow[r, "\delta"] \arrow[d, "(\phi \times_X \phi') \times 1"] & (Y \times \mathbb{R}) \times (Y' \times \mathbb{R}) \arrow[d, "(\phi \times 1) \times (\phi' \times 1)"] \\
X \times \mathbb{R} \arrow[r, "\Delta"] & (X \times \mathbb{R}) \times (X \times \mathbb{R})
\end{tikzcd}$$

$$\begin{tikzcd}
U_f \times_{X \times \mathbb{R}} V_{f'} \arrow[r, "\delta"] \arrow[d, "(u_f) \times_{X \times \mathbb{R}} (v_{f'})"] & U_f \times V_{f'} \arrow[d, "u_f \times v_{f'}"] \\
X \times \mathbb{R} \arrow[r, "\Delta"] & (X \times \mathbb{R}) \times (X \times \mathbb{R})
\end{tikzcd}$$

These fibre diagrams satisfy the hypothesis of the subsequent base change Lemma 9, which we now apply. For instance, from the Lemma and the first diagram, we conclude

$$\Gamma(X, u_{f*}\Bbbk \otimes^! (\phi' \times 1)_*\Bbbk) \cong \mathrm{H}^*(\{(y, y', z) \mid z < f(x, y)\}, \Bbbk)[-\dim X - 1] \cong \mathrm{H}^*(Y \times_X Y', \Bbbk)[-\dim X - 1]$$

where the second equality comes from just projecting out the $\mathbb{R}$ factor.

From the second diagram, and the observation

$$U_f \times_{X \times \mathbb{R}} V_{f'} = \{(y, y', z) \mid f(y) < z < f'(y')\} \subset Y \times_X Y' \times \mathbb{R}$$

we conclude

$$\begin{aligned}
\Gamma(X \times \mathbb{R}, u_{f*}\Bbbk \otimes^! v_{f'*}\Bbbk) &\cong \mathrm{H}^*(\{(y, y', z) \mid f(y) < z < f'(y')\}, \Bbbk)[-\dim X - 1] \\
&\cong \mathrm{H}^*(w^{-1}(0, \infty), \Bbbk)[-\dim X - 1]
\end{aligned}$$

The second identification comes from observing that forgetting the z coordinate is a fibration in open intervals over $w^{-1}(0, \infty)$.

Taking sections of (1) and shifting by $\dim U_f = \dim Y + 1$, we get a triangle

$$\mathrm{Hom}^*(u_{f!}\Bbbk, u_{f'!}\Bbbk) \to \mathrm{H}^*(Y \times_X Y', \Bbbk)[\dim Y - \dim X] \to \mathrm{H}^*(w^{-1}(0, \infty), \Bbbk)[\dim Y - \dim X] \xrightarrow{[1]}$$

The second morphism is evidently induced by pullback along the inclusion, which identifies the Hom space with the relative cohomology. This completes the proof. $\square$

We used the following in the proof of Theorem 8:

Lemma 9. *Let $m : M \to B$ and $m' : M' \to B$ be smooth (not necessarily proper) submersions of manifolds. Then*
$$m_* \Bbbk \otimes^! m'_* \Bbbk = (m \times_B m')_* \Bbbk[-\dim B],$$
and in particular,
$$\Gamma(B; m_* \Bbbk \otimes^! m'_* \Bbbk) = \Gamma(M \times_B M'; \Bbbk)[-\dim B].$$

Proof. By definition, $m_* \Bbbk \otimes^! m'_* \Bbbk$ is the shriek pullback along the diagonal of $m_* \Bbbk \boxtimes m'_* \Bbbk = (m \times m')_* \Bbbk$ on $B \times B'$. We write $\mathbb{D}$ for the dualizing sheaf. Consider the fibre product diagram
$$\begin{tikzcd} M \times_B M' \arrow[r, "\delta"] \arrow[d, "m \times_B m'"] & M \times M' \arrow[d, "m \times m'"] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

By base change, $\Delta^! (m \times m')_* \Bbbk_{M \times M'} = (m \times_N m')_* \delta^! \Bbbk_{M \times M'}$. Since $M \times M'$ is a manifold, $\Bbbk_{M \times M'} = \mathbb{D}_{M \times M'}[-\dim M - \dim M']$. So $\delta^! \Bbbk_{M \times M'} = \mathbb{D}_{M \times_B M'}[-\dim M - \dim M']$. But since m, m' are submersions, also $M \times_B M'$ is a manifold, and thus $\mathbb{D}_{M \times_B M'}[-\dim M - \dim M'] = \Bbbk_{M \times_B M'}[-\dim B]$. □

Remark. The idea to study the critical points of this function w goes back at least to Viterbo [Vit]. One sees easily that they correspond to Reeb chords between Λ_F and $\Lambda_{F'}$, with critical values given by integrating the contact form. If $\Lambda_F = \Lambda_{F'}$, then also the length zero chords appear: an entire copy of Λ_F. The cohomology of $\mathrm{H}^*(Y \times_X Y', w^{-1}(0, \infty); \Bbbk)$ is called the generating function cohomology [Tra2, Pus2, FR], and its use is a well established technique in the study of Legendrian knots. Different authors have preferred slightly different variants, which are related variously by excision, Poincaré duality, and taking homology or cohomology. We note that in fact the above proof is implicitly using these variants as well, in the form of invoking the open-closed exact triangle of sheaves and the Verdier duality, which are the sheaf-theoretic incarnations of excision and Poincaré duality, respectively.

4. Morse complex sequences and reconstruction of generating families

We now restrict ourselves to the setting where X is one dimensional. This is the most well studied case, being relevant for Legendrian knots and links on the one hand, and for the classical pseudo-isotopy versus isotopy question on the other. In this setting, Henry and Rutherford have developed a combinatorial abstraction of generating families, which they call Morse complex sequences [Pus2, FR, Hen, HR, HR2]. As we will explain here, this combinatorial abstraction is most naturally understood as an explicit specification of a sheaf.

4.1. **The Morse complex category.** We begin in the setting where X is zero dimensional, as this will be the basic building block for the one dimensional case. Consider a Morse function $f : Y \to \mathbb{R}$, and a metric g on Y. We ask that this pair be Morse-Smale, i.e.,

(1) f is Morse

(2) Each critical value has only one critical point
(3) Intersections of stable gradient flow cells with the unstable cells are transverse

We often suppress g from the notation. Consider the Morse complex $Morse(Y, f; \Bbbk)$, along with its filtration by critical values, $V_{<z}Morse(Y, f; \Bbbk)$. This filtration translates into a sheaf $\widetilde{Morse}(Y, f; \Bbbk)$, characterized stalkwise by $\widetilde{Morse}(Y, f; \Bbbk)_z = V_{<z}Morse(Y, f; \Bbbk)$, and related to our constructions here by

$$\underline{u}_{f!}\Bbbk \cong \widetilde{Morse}(Y, f; \Bbbk)$$

The $\mathbb{R}$ filtration $V_{<z}$ has only finitely many nontrivial steps, corresponding to the critical values. We form a more combinatorial object by recording only these nonvanishing steps. We number the critical points $\{1, \ldots, n\}$ numbered in decreasing order of critical value – the point with largest critical values is numbered 1 and the point with the smallest critical value is numbered n. We record the Morse indices in a function $\mu : \{1, \ldots, n\} \to \mathbb{Z}$. The vector space underlying the Morse complex is:

Definition 10. For $\mu : \{1 \ldots n\} \to \mathbb{Z}$, we write $\not\mu$ for the free graded $\Bbbk$-module with basis $|1\rangle, \ldots, |n\rangle$ where $\deg |i\rangle = -\mu(i)$, and decreasing filtration ${}^k\not\mu := \mathrm{Span}(|n\rangle, \ldots, |k+1\rangle)$. That is,

$${}^0\not\mu = V, \;\; {}^1\not\mu = \mathrm{Span}(|n\rangle, \ldots, |2\rangle), \quad \cdots \quad {}^{n-1}\not\mu = \mathrm{Span}|n\rangle, \;\; {}^n\not\mu = 0.$$

We now define a category whose objects are Morse complexes, and whose morphisms are set up to match $\mathrm{Hom}(\underline{u}_{f!}\Bbbk, \underline{u}_{f!}\Bbbk)$.

Definition 11. Fix an integer n and $\mu : \{1, \ldots, n\} \to \mathbb{Z}$. We define $MC_\mu(\Bbbk)$ to be the dg category with:

- Objects: square-zero operators d on $\not\mu$, which preserve the filtration on $\not\mu$ and are degree 1 with respect to the grading on $\not\mu$.
- Morphisms: $\mathrm{Hom}_{MC(\mu)}(d, d')$ is $\mathrm{Hom}_{filt}(\not\mu, \not\mu)$ as a graded vector space; i.e., it consists of the linear, filtration preserving maps $\not\mu \to \not\mu$ and carries the usual grading of a Hom of graded vector spaces. Only its differential depends on d, d', and is
$$D\phi = d' \circ \phi - (-1)^{|\phi|}\phi \circ d.$$
- Composition: usual composition of maps.

That is, we allow maps $|j\rangle\langle i|$ for $i \leq j$, i.e. lower triangular matrices, and

$$\deg |j\rangle\langle i| = \deg |j\rangle - \deg |i\rangle = \mu(i) - \mu(j)$$

and the differential is $D(|i\rangle\langle j|) = d'|i\rangle\langle j| - (-1)^{\mu(i)-\mu(j)}|i\rangle\langle j|d$.

Definition 12. Let $f : Y \to \mathbb{R}$ be a Morse function and g an appropriate metric on Y. Assume that critical points have distinct critical values. We order the n critical values, $z_1 > z_2 > \cdots > z_n$. We take $\mu(i)$ to be the Morse index of the critical point at z_i. We write $m(Y, f; \Bbbk)$ for the element of

$MC_\mu(\mathbb{k})$ obtained by writing $|i\rangle$ for the generator corresponding to the critical point whose critical value is z_i.

Recall we have a category $Gen(pt;\mathbb{k})$ whose objects are generating families over a point (i.e. just functions), and whose categorical structure comes from just pulling back that of the sheaf category under the map $f \mapsto \underline{u}_{f!}\mathbb{k}$. We will introduce a Morse-theoretic / combinatorial analogue.

Definition 13. Fix a list Λ of real numbers $z_1 > z_2 > \cdots > z_n$ and a function $\mu : \{1,\ldots,n\} \to \mathbb{Z}$. We write $GenMC_{\Lambda,\mu}(\mathbb{k})$ for the (dg) category with

- Objects: tuples of: a space Y, a Morse function $f : Y \to \mathbb{R}$, and a compatible metric g on Y so that the corresponding gradient flow is Morse-Smale, and such that f has n critical points with critical values $z_1,\ldots,z_n$ and Morse indices given by μ.
- Morphisms, differential, and composition: defined by pulling back the following categorical structures under the map $(Y,f,g) \mapsto m(Y,f;\mathbb{k})$.

Remark. The list of real numbers can be identified with a zero-dimensional Legendrian in $J^1(pt) = \mathbb{R}$, and μ is to be understood as a Maslov potential on it.

Proposition 14. *We have a commutative diagram of categories:*

$$\begin{tikzcd} GenMC_{\Lambda,\mu}(\mathbb{k}) \arrow[r] \arrow[d] & Gen_{\Lambda,\mu}(pt;\mathbb{k}) \arrow[d] \\ MC(\mu;\mathbb{k}) \arrow[r] & Sh_{\Lambda,\mu}(\mathbb{R};\mathbb{k})_0 \end{tikzcd}$$

All maps are fully faithful

Proof. First let us clarify the structure of what is being asserted. Given a map from a set A to the objects of a category B, we can upgrade A to a category by setting $\mathrm{Hom}_A(a_1,a_2) := \mathrm{Hom}_B(\pi(a_1),\pi(a_2))$; in this case the map $A \to B$ becomes a fully faithful functor. This is precisely how we have defined the vertical maps. in the statement of the proposition.

Now suppose in general given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi"] & A' \arrow[d, "\pi'"] \\ B \arrow[r, "\phi"] & B' \end{tikzcd}$$

in which B, B' are categories, ϕ is a functor, A, A' are sets and the maps π, π' carry the sets A, A' to the objects of B, B', and a set map $f : A \to A'$ is specified, together with, for each object $a \in A$, an isomorphism $I : \phi \circ \pi(a) \cong \pi' \circ f(a)$ of objects of B'.

We upgrade A, A' to categories as before, and upgrade f to a functor by the formula

$$\begin{aligned}
\mathrm{Hom}_A(a_1, a_2) &= \mathrm{Hom}_B(\pi(a_1), \pi(a_2)) \\
&\xrightarrow{\phi} \mathrm{Hom}_{B'}(\phi \circ \pi(a_1), \phi \circ \pi(a_2)) \\
&\xrightarrow{I} \mathrm{Hom}_{B'}(\pi' \circ f(a_1), \pi' \circ f(a_2)) \\
&= \mathrm{Hom}_{A'}(f(a_1), f(a_2))
\end{aligned}$$

Evidently I now becomes a natural isomorphism of functors.

We now return to the case at hand. It remains only to name, at the level of sets, the arrow at the bottom of the diagram; show it is fully faithful, and then to argue that the diagram commutes in the sense described above. This arrow is: first use Λ to specify the location of the non-trivial steps in an $\mathbb{R}$-filtered complex which otherwise is the filtered complex $\not\mu$, i.e., so that $m(Y, f; \mathbb{k})$ turns into $Morse(Y, f; \mathbb{k})$, and then second, turn this $\mathbb{R}$-filtered complex into a sheaf, i.e. such that $Morse(Y, f; \mathbb{k})$ becomes $\widetilde{Morse}(Y, f; \mathbb{k})$ as discussed in the introduction. The first step is obviously fully faithful, and the definition on morphisms and full faithfulness of the second step is the equivalence of $\mathbb{R}$-filtered complexes with certain sheaves on $\mathbb{R}$. (We explain this at greater length, albeit factored through the representation theory of the A_n quiver, in [NRSSZ, Sec. 7.3].)

The assertion that the diagram commutes on objects is the statement that the Morse complex computes cohomology, in the sheaf theoretic form $\widetilde{Morse}(Y, f; \mathbb{k}) \cong u_{f!}\mathbb{k}$ explained in the introduction. $\square$

Remark. Assume $\mathbb{k}$ is a field. One view on the Morse complex category is that it is defined so that the isomorphisms of Morse complexes are precisely those considered by Barranikov in his classification of Morse complexes over a field. The Morse complex associated to a generating function necessarily has positive degree generators, hence the indexes μ must all be positive; aside from this restriction it is easy to construct functions with Morse complexes in each Barranikov equivalence class. That is, for $\mu \geq 0$, the morphism $GenMC_{\Lambda,\mu}(\mathbb{k}) \to MC(\mu; \mathbb{k})$ is an equivalence. If Y is a vector space, and $f : Y \to \mathbb{R}$ required to be linear at infinity, the corresponding Morse complex is evidently acyclic, but again this is the only restriction.

Under the assumption that $\mathbb{k}$ is a field, we have shown in [NRSSZ, Sec. 7.3] that $MC(\mu; \mathbb{k}) \to Sh_{\Lambda,\mu}(\mathbb{R}; \mathbb{k})_0$ is also an equivalence. Combining these observations, it follows that every object in $Sh_{\Lambda,\mu}(\mathbb{R}; \mathbb{k})_0$ with acyclic stalks at $\pm\infty$ arises from a generating function.

4.2. **Morse complex sequences.** Our goal now is to set up a similar diagram in the setting of one-parameter families of functions. In fact we will restrict ourselves to $X = \mathbb{R}$. To disambiguate the two copies of the real line, we write our generating family as $\mathbb{R}_x \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}_z$.

Recall we write Φ_f for the image of Λ_f in $J^0(\mathbb{R}_x) = \mathbb{R}_x \times \mathbb{R}_z$. This is the *front diagram* of the Legendrian knot Λ_f, and also the *Cerf graphic* of the family of functions f_x [HW]. At any x satisfying the first two conditions above, $\Phi_f \cap \{(x - \epsilon, x + \epsilon)\} \times \mathbb{R}_z$ consists of smooth nonintersecting paths projecting diffeomorphically to $\mathbb{R}_x$.

Remark. I think of the sheaf $u_{f!}\Bbbk$ as a sort of further decoration of the Cerf graphic.

A generating family $\mathbb{R}_x \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}_z$, together with the data of a family of metrics g, is said to be Morse-Cerf if f_x only fails to be Morse-Smale at finitely many x, and at these points satisfies one of the following:

- The function f_x fails to be Morse at a single point, where its singularity is locally that at zero of the function $x_1^3 + \sum_{i>1} \pm x_i^2$. The critical values of this and all other critical points are distinct. The corresponding front diagram near the corresponding critical value is an ordinary cusp. This is termed a birth or death singularity, according as the cusp is $\prec$ or $\succ$.
- The function f_x is Morse, but two critical points have coincident critical values. This appears as a crossing in the front diagram.
- The function f_x is Morse, all critical points have distinct critical values, but there is exactly one pair p, p' with non-transversely intersecting stable and unstable cells. Moreover, p and p' have the same Morse indices. This sort of singularity is called a handle slide, and is not visible from the front diagram.
- The function f_x is Morse, all critical points have distinct critical values, but there is exactly one pair p, p' with non-transversely intersecting stable and unstable cells. Moreover, p and p' have Morse indices differing by one. This sort of singularity is not visible from the front diagram, and corresponds to the collisions of algebraically canceling gradient trajectories. As such it does not affect the sheaf $\underline{u}_{f!}\Bbbk$, so we ignore such events.

It is known that a generic perturbation of any generating family is Morse-Cerf [Cer, Lau].

In the usual Morse theory, a Morse-Smale function determines a cell decomposition of the manifold, and the attaching maps can be characterized by certain gradient flow trajectories. In fact, one can give cell decompositions for the more general sort of functions detailed above; this is done in [Lau]. We introduce, essentially following [HR2] at the level of global objects, a combinatorial structure to organize together all this generalized Morse theory. The structure takes the form of a constructible sheaf of dg categories on $\mathbb{R}_x$.

Definition 15. Let Λ be a Legendrian with a generic front diagram, equipped with a Maslov potential μ. Let $\mathcal{S}$ be a stratification of $\mathbb{R}_x$, such that the zero dimensional strata include at least the x-coordinates of the singularities of the front projection Λ. We will define a sheaf $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)$ of dg categories on $\mathbb{R}_x$, constructible with respect to the stratification $\mathcal{S}$.[3] To do this, it suffices to give the stalk on each stratum, together with left and right generization maps at points on the zero dimension strata.

At any point x in a 1-dimensional stratum (a, b), we take $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)_x := MC_{\mu_x}(\Bbbk)$. We view these categories for various $x \in (a, b)$ as canonically identified, and denote them in common as $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)_{(a,b)}$.

[3]Strictly speaking, these words should be interpreted with appropriately higher categorical sophistication. We do not emphasize this point here because it is essentially irrelevant in the setting at hand, due to the lack of topology of the base $X = \mathbb{R}_x$, even after stratification.

On a 0-dimensional stratum over $x \in \mathbb{R}_x$, we will in each case define an object of MCS_x to be built from data including elements $d_{x-\epsilon} \in MC_{\mu_{x-\epsilon}}(\Bbbk)$ and $d_{x+\epsilon} \in MC_{\mu_{x+\epsilon}}(\Bbbk)$; the generization maps will just return these elements. Specifically, at a point

- ... *over which the front projection is nonsingular*, $\mu_{x-\epsilon} = \mu_x = \mu_{x+\epsilon}$, so we can canonically identify $MC(\mu_{x-\epsilon}) = MC(\mu_x) = MC(\mu_{x+\epsilon})$. An element of $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)_x$ will be by definition a triple $d_{x-\epsilon} \xleftarrow{\sim} d_x \xrightarrow{\sim} d_{x+\epsilon}$, and we define morphisms in MCS_x to be triples of morphisms commuting with this structure. Note that the forgetful map to any of the three pieces is an equivalence of categories.
- ... *over which the front projection suffers a crossing between the ith and $i+1$st strands*, we write $s_{i,i+1}$ for the permutation matrix $|i\rangle \leftrightarrow |i+1\rangle$. We take an object in our category to be a pair $(d_{x-\epsilon}, d_{x+\epsilon})$ where $d_{x-\epsilon}s_{i,i+1} = s_{i,i+1}d_{x+\epsilon}$, and a morphism to be a pair of morphisms commuting with $s_{i,i+1}$.
- ... *over which the ith and $i+1$st strands meet in a right cusp (death)*, we require that $\langle k+1|d_{x-\epsilon}|k\rangle$ is invertible. For this discussion let us change the labelling of the basis of $\mu\!\!\!/_{x+\epsilon}$ from $1,\ldots,n$ to $1,2,\ldots,k-1,k+2,\ldots,n$, for consistency with the basis of $\mu\!\!\!/_{x-\epsilon}$.

  By the invertibility assumption, the quotient $\mu\!\!\!/_{x-\epsilon}/(|k\rangle, d_{x-\epsilon}|k\rangle)$ is freely generated by $|i\rangle$ for $i \neq k, k+1$. Thus there is a unique map

  $$\begin{array}{rcll} \pi_x : \mu\!\!\!/_{x-\epsilon} & \to & \mu\!\!\!/_{x+\epsilon} & \\ |k\rangle & \mapsto & 0 & \\ d_{x-\epsilon}|k\rangle & \mapsto & 0 & \\ |i\rangle & \mapsto & |i\rangle & \text{for } i \neq k, k+1 \end{array}$$

  We take the objects of MCS_x to be the pairs $(d_{x-\epsilon}, d_{x+\epsilon})$ which commute with π_x; likewise for morphisms.
- ... *over which the ith and $i+1$st strands meet in a left cusp (birth)*, we define the category as in the previous case.

The sections of the sheaf of categories $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)$ on a given open set U have an explicit description as a limit (this is what the word sheaf means). Specifically, if $U = (x_0, x_{n+1})$ and the zero dimensional strata in this region are $x_1 < x_2 < \cdots < x_n$, then

$$MCS_{\Lambda,\mathcal{S}}(U) = \lim\left(\prod MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)_{x_i} \rightrightarrows \prod MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)_{(x_j,x_{j+1})}\right)$$

Note that since it is a sheaf of *categories* rather than of sets, an object of the category described by this limit is a diagram of the form

$$S_{(x_0,x_1)} \leftarrow S_{x_1} \rightarrow S^L_{(x_1,x_2)} \cong S^R_{(x_1,x_2)} \leftarrow S_{x_2} \rightarrow S^L_{(x_2,x_3)} \cong S^R_{(x_2,x_3)} \cdots$$

where $S_\bullet \in MCS_\bullet$. Since for any $x \in (x_i, x_{i+1})$ we identify, by definition, $MCS_x = MCS_{(x_i,x_{i+1})}$, I prefer to write the above as:

$$S_{x_1-\epsilon} \leftarrow S_{x_1} \rightarrow S_{x_1+\epsilon} \cong S_{x_2-\epsilon} \leftarrow S_{x_2} \rightarrow S_{x_2+\epsilon} \cong S_{x_3-\epsilon} \cdots$$

That is, when computing sections in a sheaf of categories, objects can be glued along isomorphisms, not just along identities.

In fact, we can remove all handle slide singularities:

Proposition 16. *Let Λ be a knot, $\mathcal{S}$ a stratification of $\mathbb{R}_x$ in which the zero dimensional strata include all the x-coordinates of the images of the singularities of the front projection of Λ. Let x_0 be a zero dimensional stratum which is* not *the x-coordinate of a singularity, and let x_{-1} and x_1 be the immediately preceeding and succeeding zero dimensional strata. Let $\mathcal{S}'$ be the stratification in which the strata $(x_{-1}, x_0), x_0, (x_0, x_1)$ are merged into one stratum (x_{-1}, x_1).*

Then there is an equivalence of (sheaves of) categories $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk) \to MCS_{\Lambda,\mu,\mathcal{S}'}(\Bbbk)$

Proof. The equivalence will be the identity outside of $[x_{-1}, x_1]$. In this region, we map an object

$$S_{x_{-1}} \to S_{x_{-1}+\epsilon} \xrightarrow{b} S_{x_0-\epsilon} \xleftarrow{c} S_{x_0} \xrightarrow{d} S_{x_0+\epsilon} \xrightarrow{e} S_{x_1-\epsilon} \leftarrow S_{x_1}$$

to

$$S_{x_{-1}} \to S_{x_{-1}+\epsilon} \xrightarrow{edc^{-1}b} S_{x_1-\epsilon} \leftarrow S_{x_1}$$

which makes sense since b, c, d, e are all isomorphisms. We map the morphism spaces accordingly. $\square$

Remark. Note that we may begin with a Morse complex sequence in which all "gluing isomorphisms" are identities, and end up with one in which this is no longer the case, i.e. we may begin above in a situation where b, e are both the identity, and then end up with a non-identity gluing map dc^{-1}. Conversely, by subdividing, we can represent any Morse complex sequence as one with all gluing isomorphisms the identity.

If $\mathcal{S}'$ refines $\mathcal{S}$, there is a natural fully faithful inclusion $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk) \to MCS_{\Lambda,\mu,\mathcal{S}'}(\Bbbk)$ given by setting the stalk of the sheaf at any new zero dimensional strata (which are necessarily beneath smooth points of the front projection) to triples of the form $d_{x-\epsilon} \xleftarrow{=} d_x \xrightarrow{=} d_{x+\epsilon}$. Proposition 16 implies that all these inclusions are equivalences. Thus we write $MCS_{\Lambda,\mu}(\Bbbk) = \lim MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk)$ for the direct limit; the natural maps $MCS_{\Lambda,\mu,\mathcal{S}}(\Bbbk) \to MCS_{\Lambda,\mu}(\Bbbk)$ are all equivalences.

As we mentioned already, the original purpose of the Morse complex sequences was to record the Morse-theoretic information in a generating family over the line. That is:

Proposition 17. *Let $\mathbb{R}_x \xleftarrow{\phi} Y \xrightarrow{f} \mathbb{R}_z$ be Morse-Cerf with respect to some family of metrics. Let $\mathcal{S}$ be the stratification of $\mathbb{R}_x$ in which the zero-dimensional strata are the x for which f_x is not Morse-Smale. Let μ be the Maslov potential on the Legendrian generated by Λ given by recording the Morse indices of the corresponding critical points. Then there is a Morse complex sequence $\mathfrak{p}(f) \in MCS_{\Lambda,\mu,\mathcal{S}}$ characterized by:*

- $\mathfrak{p}(f)_x = m(Y, f_x; \Bbbk)$ *for x outside the zero dimensional strata*
- *For x beneath a handle slide singularity of f, the morphism $\phi_x : m_{x-\epsilon} \xrightarrow{\sim} m_{x+\epsilon}$ is $\phi_x = 1 \pm |j\rangle\langle i|$.*

- *All gluing isomorphisms are the identity.*

Proof. The proposition as stated appears in [HR]; see also [FR, Sec. 5.1.2], both of which draw from [Lau]. □

As with $Gen, GenMC$, we use this map on objects to define the structure of a category on the Morse-Cerf generating families.

Definition 18. Fix $\Lambda \subset J^1(\mathbb{R}_x)$ in general position, let μ be a Maslov potential on Λ. We write $GenMCS_{\Lambda,\mu}(\Bbbk)$ for the (dg) category whose objects are Morse-Cerf generating families which generate Λ with corresponding Morse indices given by μ, and the rest of whose categorical structure is defined by pulling back along $f \mapsto \mathfrak{p}(f)$.

On the other hand, a Morse complex sequence can be used to specify a sheaf. From an object $S \in MCS_{\Lambda,\mu,S}(\Bbbk)$, we define a sheaf $\widetilde{S}$ as follows. Above any $x \in \mathbb{R}_x$ not in a zero dimensional stratum of the stratification, we have $S_x \in MC_{\mu_x}(\Bbbk)$ and can use the morphism $MC_{\mu_x}(\Bbbk) \to Sh_{\Lambda_x,\mu_x}(x \times \mathbb{R}_z, \Bbbk)$ to define the sheaf $\widetilde{S}|_{x \times \mathbb{R}_z}$. At points beneath a handle slide, the same prescription applies. When x sits beneath a crossing, the specification of $\widetilde{S}_{x,z}$ can be taken unambiguously to be identical to that of $\widetilde{S}_{x \pm \epsilon,z}$ except at the coordinates of the crossing itself. Here, we take $\widetilde{S}_{x,z} = \widetilde{S}_{x \pm \epsilon, z-\delta}$. Finally, for x beneath a right cusp, we define $\widetilde{S}_{x,z} = \widetilde{S}_{x-\epsilon,z}$, except at the cusp point; the rightward generization maps are given by the appropriate filtered piece of π_x. At the cusp point itself, we set $\widetilde{S}_{x,z} = \widetilde{S}_{x-\epsilon,z-\delta}$. By construction, $\widetilde{S} \in Sh_{\Lambda,\mu}(\mathbb{R}_x \times \mathbb{R}_z, \Bbbk)_0$.

Proposition 19. [NRSSZ] *The map on objects above underlies a fully faithful (dg) functor*

$$\mathfrak{r} : MCS_{\Lambda,\mu,S}(\Bbbk) \to Sh_{\Lambda,\mu}(\mathbb{R}_x \times \mathbb{R}_z, \Bbbk)_0$$

This functor is an equivalence at least when $\Bbbk$ is a field and Λ is in plat position.

Proof. This is proven in [NRSSZ, Sec. 7.3]. The basic strategy of proof is to observe that both sides are sheaves of categories over the line $\mathbb{R}_x$, so it suffices to define the morphism and prove the theorem locally; i.e., over each neighborhood $(x - \epsilon, x + \epsilon)$. That reduces things to a case study of the picture in each of: the nonsingular setting, a crossing, a cusp. The restriction to field coefficients is due to the fact that in the course of the proof, the sheaf category is given a quiver description, and we invoke Gabriel's theorem. □

Remark. In [NRSSZ] the Morse complex sequence category is defined slightly differently. There, we did not emphasize the possibility of zero dimensional strata over which the front projection is nonsingular – i.e., the formal handle slides – although we did not exclude them either. Also, we allowed more general objects at the crossing. This difference is entirely cosmetic: it is easy to see that category of a crossing object in [NRSSZ] is equivalent to the category over an interval here which contains a crossing object, and a formal handle slide on each side of it. The point is that the Morse complex category is defined in such a way that it can always absorb the formal handle slides as in Proposition 16, this absorbtion commutes with the map to Sh. We have allowed the

formal handle slides to appear explicitly here, and restricted the form of the crossing, just so as to better reflect the Morse theory. The opposite choice was made in [NRSSZ] so as to be closer to the augmentation category.

We also note that the above proposition can be proven without assuming the knot is in plat position; we do not do so because for all our results we can and will work after putting the knot in plat position. In fact, the more general form of the above proposition follows from the result in plat position and the results proven here, since both generating families and sheaves are known to transform naturally under Legendrian isotopy.

Proposition 20. *We have a commutative diagram of categories:*

$$\begin{tikzcd} GenMCS_{\Lambda,\mu}(\Bbbk) \arrow[r] \arrow[d, "\mathfrak{p}"] & Gen_{\Lambda,\mu}(\mathbb{R}_x;\Bbbk) \arrow[d] \\ MCS_{\Lambda,\mu}(\Bbbk) \arrow[r, "\mathfrak{r}"] & Sh_{\Lambda,\mu}(\mathbb{R}_x \times \mathbb{R}_z;\Bbbk)_0 \end{tikzcd}$$

All maps are fully faithful.

Proof. The vertical arrows are fully faithful by definition. The bottom arrow we discussed above. The top arrow, on objects, just forgets the metric. The diagram commutes on objects in the sense that $\mathfrak{r} \circ \mathfrak{p}(f) \cong u_{f!}\Bbbk$; this can be checked locally on $\mathbb{R}_x$ by generalized Morse theory. More precisely, what is being asserted here is that the restriction map from the $\mathbb{R}$-filtered cohomology of the neighborhood of the fibre over x to the $\mathbb{R}$-filtered cohomology of the fibres over $x_{\pm\epsilon}$ is given precisely by the restriction maps specified in the Morse complex sequence category. This fact is what is checked in [FR, Sec. 5.1.2] or [HR2], drawing from [Lau]. We use this equivalence to define the top arrow as a morphism of categories, as in Proposition 14. □

4.3. **Surjectivity.** We turn to proving Theorem 5. First we collect results from the literature which ensure that certain Morse complex sequences can be realized by functions. We refer to zero dimensional strata which do not lie beneath a singularity as "formal handle slides".

Proposition 21. *Let Λ be a Legendrian knot or link, and μ a Maslov potential. Let $S \in MCS_{\Lambda,\mu}(\mathbb{Z})$ be a Morse complex sequence for which:*

- *Morse complexes near a crossing between the i'th and $i+1$'st strands have $\langle i+1|d|i\rangle = 0$*
- *Morse complexes near a cusp between the i'th and $i+1$'st strands have $\langle i+1|d|i\rangle = 1$*
- *The isomorphisms ϕ_x characterizing formal handle slides take the form $\phi_x = 1 \pm |j\rangle\langle i|$*

Let M be simply connected, and let $f_{-\infty} : M \to \mathbb{R}_z$ be a Morse function with Morse complex $S_{-\infty}$. Assume $2 \leq \mu \leq \dim M - 2$. Then there exists a generating family $\mathbb{R}_x \xleftarrow{\phi} \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z$ extending $f_{-\infty}$ such that $\mathfrak{p}(\mathbb{R}_x \xleftarrow{\phi} \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z) = S$.

If M is a linear space, the front projection is compactly supported, and the Morse function $f_{-\infty}$ is linear, then the generating family can be chosen to be linear at infinity.

Proof. We build the function from the left to the right along the x line. At the left, the function is given. Varying the function to create the left cusps is elementary. Before a crossing we have $\langle i+1|d|i\rangle = 0$. This says that numerically, the stable and unstable cells of the corresponding critical points do not intersect. If the cells *actually* did not intersect, then the the "independent trajectories principle" would say that we can deform the function as desired [HW, Chap. 1, Sec. 7]. But since the space is simply connected and $2 \leq i < i+1 \leq \dim M - 2$, we actually cancel the numerically cancelling trajectories, as in the proof of [Mil, Thm. 6.4, p.49]. The desired formal handle slides can be realized by the classical result on existence of handle slides, see e.g. [HW, Chap. 2, Lemma 1.2]. Again this requires $2 \leq \mu \leq \dim M - 2$. Finally, before the right cusps we have $\langle i+1|d|i\rangle = 1$; again if this signified the existence of an actual single gradient trajectory between the critical points we could cancel them by the flushing lemma [Mil, Thm. 5.4, p. 49]. But in fact assumptions on the space allow us to actually cancel the numerically cancelling trajectories, as asserted by [Mil, Thm. 6.4, p.49], which too requires $2 \leq \mu \leq \dim M - 2$. Thus, the function can be deformed to create a death singularity in the desired way.

Finally, under the hypothesis that the front projection is compactly supported, M is linear, and $f_{-\infty}$ is linear, note that we may make all the above changes in some compact region, and that after the right cusps, we have a function with no critical points, which can be deformed to a linear function as desired. □

It remains to show that all Morse complex sequences can be put in the form required by Proposition 21. So fix a Morse complex sequence S. First, subdivide all one-dimensional strata, and replace S with a sequence in which all gluing isomorphisms are identity maps, as in the remark below Proposition 16. The left and right generization maps at a formal handle slide are lower triangular isomorphisms. We want to have all formal handle slides given by unipotent lower triangular isomorphisms (i.e. with 1s on the diagonal). So we factor out a diagonal matrix and begin pushing it to the right. Since unipotent matrices are normal in the group of lower triangular matrices, and since conjugation by a transposition preserves diagonal matrices, we can push it past any handle slides or crossings. Upon reaching a cusp, the relevant piece of the diagonal matrix can be absorbed into an automorphism of the cusp. We have now ensured that all formal handle slides are unipotent. By row reduction, we can express any unipotent lower triangular matrix as a product of elementary matrices; by further subdivision we can implement this factoring and thus ensure that all of our formal handle slides are characterized by isomorphisms of the form $\phi_x = 1 \pm |j\rangle\langle i|$. This completes the proof of Theorem 5.

Remark. There is a very convenient graphical notation for describing and manipulating Morse complex sequences; see [Hen, HR]. In those papers, a notion of equivalence of Morse complex sequences is introduced as generated by certain local moves; we leave it to the reader to check directly that these moves give isomorphisms in the Morse complex sequence category. Abstractly this is true because locally these moves correspond to considering a fixed generating family f but varying the metric; since isomorphism in the Morse complex sequence category only depends

on the sheaf $\underline{u}_{f!}\Bbbk$, hence not on the metric, such moves give isomorphisms of Morse complex sequences. These local moves can be used to simplify a Morse complex sequence *much* further than we have done above. In fact it can be ensured that the only appearing handle slides have $\phi_x = 1 \pm |i\rangle\langle i+1|$ where $i, i+1$ name strands which are just about to cross. Such a Morse complex sequence is said to be in A-form; see e.g. [HR, Def. 7.1].

Remark. Let us spell out more precisely the relation of the above discussion to the proof of the h-cobordism theorem [Sma1, Sma2, Mil]. Recall this result deals with the following situation: one has a cobordism M from M_- to M_+ such that the inclusion of either end is a homotopy equivalence; one wants to know whether this cobordism is trivial. If it is trivial, projection $M = M_- \times I \to I$ gives a Morse function without critical points, and conversely such a function gives M the structure of a trivial cobordism.

The idea of the proof of the h-cobordism theorem is to take an arbitrary Morse function $f_{-\infty} : (M, M_-, M_+) \to (\mathbb{R}_z, -\infty, \infty)$, and evolve it into a critical-point-free Morse function f_∞. This evolution can be viewed as a family of functions: $\mathbb{R}_x \xleftarrow{\phi} \mathbb{R}_x \times M \xrightarrow{f} \mathbb{R}_z$. Note that the assertion that f_∞ has no critical points is the same as the assertion that $\underline{u}_{f!}\Bbbk$ is acyclic when $x \gg 0$.

One way to proceed would be the following: first build a sheaf $\mathcal{F}$ on $\mathbb{R}_x \times \mathbb{R}_z$ with $\mathcal{F}|_{-\infty \times \mathbb{R}_z} = \underline{u}_{f_{-\infty}!}\Bbbk$ and $\mathcal{F}|_{\infty \times \mathbb{R}_z}$ everywhere acyclic, then extend the function $f_{-\infty}$ to a function f such that $\underline{u}_{f!}\Bbbk \cong \mathcal{F}$.

Building the sheaf $\mathcal{F}$ is the easy part. For instance, suppose the Morse complex $Morse(M, f_{-\infty}; \Bbbk)$ admits a Barranikov normal form. This is just the same as saying that the sheaf $u_{f_{-\infty}!}\Bbbk$ is the direct sum of n copies of a rank one constant sheaf supported on a half-open interval, for various half-open intervals. Each such summand is the restriction to the midline of the unique microlocal rank one sheaf on the standard unknot, so we define $\mathcal{F}$ as the direct sum of n copies of (the right half of) this sheaf.

It remains to find f. Assuming the Morse indices of all the critical points of f_∞ satisfy $2 \le \mu \le \dim M - 2$, then the argument we have used to prove Theorem 5 gives such a function. This is a rephrasing of part of the original proof of the h-cobordism theorem; the remaining part is about how to reduce to the above situation in the first place. A careful study of this part of the proof may allow our requirement $2 \le \mu$ to be relaxed.

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